

Nonlinear elastic equation of state of solids subjected to uniaxial homogeneous loading

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Applying the finite deformation theory to a solid, which possesses either cubic or isotropic symmetry at stress-free natural state and is subsequently loaded homogeneously in uniaxial direction, one obtains a stress (or strain) dependence of the Young's modulus, Poisson's ratio, and a volume (or density) change, together with a nonlinear elastic relation between stress and strain. These are all expressed in terms of the second and third order elastic constants of the solid material. These expressions are illustrated with examples of cubic silicon crystal, isotropic carbon steel, Pyrex glass, and polystyrene at the relaxed state. © 2000 Kluwer Academic Publishers

1. Introduction

Murnaghan [1, 2] developed the finite deformation theory using the second and third order elastic constants of isotropic and crystalline materials. One of his fundamental contributions is a derivation of a relation between the (Cauchy) stress and the thermodynamic (second Piola-Kirchhoff) stress. In the finite deformation theory the strength or the elastic constant of a material at a stressed state varies with stress (strain) acting on the material, which was theoretically well accounted for, based on an interatomic or intermolecular force [3]. Birch [4, 5] extended Murnaghan's work to a cubic material and derived the well-known Birch's equation of state in geology, which relates a change in density (or volume) to a (hydrostatic) pressure acting on the materials, using the parameters described in terms of the second and third order elastic constants. The work on the equation of state or the pressure-volume relationship abounds in literature [6]. Murnaghan [2] derived a nonlinear relation between the engineering stress referred to the unit area at the stress-free natural state and the engineering strain referred to the stress-free state for an initially isotropic material subjected to uniaxial homogeneous loading. Seeger and Buck [7] extended Murnaghan's treatment to a cubic material. The nonlinear field theories of mechanics including the finite deformation theory is well described in the article by Truesdell and Noll [8].

Brugger [9] gave an elegant definition of higher order elastic constants, and using his definition, Thurston and Brugger [10] derived expressions for the ultrasonic wavespeeds in stressed solid media in terms of the second and third order elastic constants. Extensive data of the second and third order elastic constants of numerous solids of various symmetry groups were compiled by Hearmon [11]. Some useful relations at finite deformation are described by Thurston [12] and thermody-

namics of crystals in reference to the stress-free state is given in detail in the book of Wallace [13]. Recently, the first author [14] has generalized the thermodynamics of elastic solids in reference to a state of arbitrary finite deformation and derived the expressions for the effective Young's modulus and Poisson's ratio of a material at a stressed state under isothermal and adiabatic conditions in terms of the second order thermodynamic elastic stiffness coefficients and three principals stresses acting on the material.

To the authors' knowledge, a nonlinear analytic relation between the Cauchy stress referred to the state of finite deformation and the strain referred to the natural state is not fully established in the case of uniaxial homogeneous loading, notwithstanding the innumerable works done on the finite deformation of materials and albeit a ubiquitous use of the tension test as a tool for material testing and characterization. This, so called the elastic equation of state that describes a behavior of material in a uniaxial tension or compression test, is analogous to the equation of state in high-pressure physics, which relates hydrostatic pressure or spherical Cauchy stress to volume change from the stress-free initial state. In this paper the authors derive not only the nonlinear relation between the Cauchy stress and strain but also the variation of the effective Young's modulus and Poisson's ratio defined at the state of finite deformation with stress or strain. These can be anticipated from the finite deformation theory but their analytical formulas have not yet been expressed to date. In addition, a change in density/volume, which includes a contribution from higher-order strain/stress terms, is given in this work. They are all expressed in terms of the second and third order elastic constants of a material. We provide a comprehensive treatment of finite deformation theory for the case of uniaxial loading especially from the thermodynamic point of view and illustrate the

nonlinear elastic behavior with four typical materials: cubic silicon, polycrystalline carbon steel, amorphous Pyrex glass, and polymeric polystyrene at the stress-free natural state.

The amount of nonlinearity as compared with that of linearity for ordinary materials is quite small below the elastic limit of a material for strain less than 1%, but it is appreciable and can be detected from precision ultrasonic wavespeed measurements [15, 16]. Most of the third-order elastic (TOE) constants of materials reported in literature have been ultrasonically measured. Understanding anharmonic properties of materials, such as the lattice vibration and the thermal expansion coefficient of solids, can be facilitated by the study of nonlinear elastic behavior and the TOE constants. For strain larger than 1% the nonlinearity term is significant and should be accounted for in the meaningful solution of elasticity problems.

For example, when fine whiskers are deformed, very large strains are present. Ruoff and his associates [17–20] have analyzed this problem using finite elasticity and found the conditions for macroscopic elastic instabilities in tension and compression of perfect crystals of diamond, silicon and germanium. The present paper extends this work. The immediate neighborhood of the interface layer of thin films that are epitaxially grown on the substrate, especially that of heteroepitaxially grown ones, may be subjected to strains as large as 10%, since the interface atoms of the thin films conform to the surface morphology of the substrate and are thus prohibited from being plastically deformed. A recent study of Lee *et al.* [21] on a polymethyl methacrylate (PMMA) film deposited on an aluminum substrate by using picosecond ultrasonics indicates that the longitudinal wavespeed in the PMMA interface layer substantially increases from that of a bulk specimen, suggesting that the PMMA interface layer may be subjected to a large strain. Strains substantially larger than 10% can be found in the deep interior of planets including the earth and inside a diamond anvil used in a laboratory, where a specimen may be under pressures well exceeding its Young's modulus at the stress-free natural state [22]. Under these extremely high deformations, understanding the physical behaviors of solids will not be possible without consideration of the nonlinear behavior of materials.

2. Description of deformation states and symbols

Consider a solid specimen of cubic or isotropic symmetry, which is in the stress-free natural state. The Cartesian coordinates of a particle in the natural state specimen is denoted by vector \mathbf{a} . The specimen undergoes an arbitrary finite homogeneous deformation under uniaxial loading in the direction 3, which coincides in the case of the cubic specimen with one of the cubic axes, say, the [001] direction. We denote the coordinates of the corresponding particle at finite deformation by \mathbf{X} , which is said to be in an initial state. The initial state is under arbitrary static stress and includes the natural state when the stress is zero. We take the coordinate axes of both the natural state \mathbf{a} and the initial state \mathbf{X} to be parallel to the directions of material

symmetry, which coincide with principal stress directions. Finally, small disturbances caused either by an isothermal small uniaxial loading or by an isentropically propagating waves are superposed on the initial state. We denote a new corresponding coordinate of the particle by \mathbf{x} .

Displacements between the various states are expressed by

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{U} = \mathbf{X} - \mathbf{a}. \quad (1)$$

An engineering strain ε_{ij} and a Lagrangian strain η_{ij} , both referred to the natural state, are respectively given by

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial a_j} + \frac{\partial U_j}{\partial a_i} \right), \quad (2)$$

$$\eta_{ij} = \frac{1}{2} \left(\frac{\partial X_m}{\partial a_i} \frac{\partial X_m}{\partial a_j} - \delta_{ij} \right) = \varepsilon_{ij} + \frac{1}{2} \frac{\partial U_m}{\partial a_i} \frac{\partial U_m}{\partial a_j}. \quad (3)$$

We also define a strain referred to the initial state

$$\begin{aligned} \xi_{ij} &= \frac{1}{2} \left(\frac{\partial x_m}{\partial X_i} \frac{\partial x_m}{\partial X_j} - \delta_{ij} \right) \\ &= \frac{1}{2} \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_m}{\partial X_i} \frac{\partial u_m}{\partial X_j} \right). \end{aligned} \quad (4)$$

A principal stretch λ_i ($i = 1, 2, 3$) from the natural state for the initial state of homogeneous deformation is given by

$$\frac{\partial X_i}{\partial a_j} = \delta_{ij} \lambda_i \quad (i \text{ not summed}). \quad (5)$$

Note that for the uniaxial homogeneous loading in the X_3 direction, $\lambda_1 = \lambda_2$ and $\eta_1 = \eta_{11} = \eta_{22} = \eta_2$, where η_i are the principal Lagrangian strains defined by $\eta_i = \eta_{ij} \delta_{ij}$ (i fixed; $i = 1, 2, 3$). The principal stretches λ_i are related to the principal Lagrangian strains η_i by

$$\eta_i = \frac{1}{2} (\lambda_i^2 - 1). \quad (6)$$

Let U and F denote the internal and free energies per unit mass, respectively. Let S and T stand for the entropy and absolute temperature, respectively. We use the superscripts/subscripts S and T to represent adiabatic and isothermal processes, respectively. The thermodynamic stresses τ_{ij}^a and τ_{ij}^X , referred respectively to the natural and initial states, are defined as [8, 23]

$$\tau_{ij}^a = \rho_a \left(\frac{\partial U}{\partial \eta_{ij}} \right)_S = \rho_a \left(\frac{\partial F}{\partial \eta_{ij}} \right)_T, \quad (7)$$

$$\tau_{ij}^X = \rho_X \left(\frac{\partial U}{\partial \xi_{ij}} \right)_S = \rho_X \left(\frac{\partial F}{\partial \xi_{ij}} \right)_T. \quad (8)$$

The thermodynamic stress τ_{ij} above is also called the second Piola-Kirchhoff stress. A (Cauchy) stress $\sigma_{ij}(\mathbf{X})$ and a $\tau_{ij}^X(\mathbf{X})$, both evaluated at the initial state \mathbf{X} , are equal to each other. It was shown by Murnaghan [1] that they are related to the thermodynamic stress $\tau_{ij}^a(\mathbf{X})$

evaluated at the initial state \mathbf{X} by

$$\sigma_{ij}(\mathbf{X}) = \tau_{ij}^X(\mathbf{X}) = \frac{\rho_X}{\rho_a} \frac{\partial X_i}{\partial a_k} \frac{\partial X_j}{\partial a_l} \tau_{kl}^a(\mathbf{X}), \quad (9)$$

which reduces in the case of the uniaxial homogenous loading to

$$\sigma_{ij}(\mathbf{X}) = \frac{1}{\lambda_1^2 \lambda_3} \lambda_i \lambda_j \tau_{ij}^a(\mathbf{X}) \quad (i, j \text{ not summed}). \quad (10)$$

We also define the thermodynamic adiabatic and isothermal elastic stiffness coefficients referred to the initial state as

$$\begin{aligned} C_{ijkl}^S &= \left(\frac{\partial \tau_{ij}^X}{\partial \xi_{kl}} \right)_S = \rho_X \frac{\partial^2 U(\xi, S)}{\partial \xi_{ij} \partial \xi_{kl}}, \\ C_{ijkl}^T &= \left(\frac{\partial \tau_{ij}^X}{\partial \xi_{kl}} \right)_T = \rho_X \frac{\partial^2 F(\xi, T)}{\partial \xi_{ij} \partial \xi_{kl}}. \end{aligned} \quad (11)$$

The internal energy under adiabatic condition and the free energy under isothermal condition can be expanded in terms of the Taylor series about the natural state and they are expressed as

$$\begin{aligned} \rho_a U(\eta, S) &= \rho_a U(0, S) + \left(\frac{1}{2} \right) c_{ijkl}^S \eta_{ij} \eta_{kl} \\ &+ \left(\frac{1}{6} \right) c_{ijklmn}^S \eta_{ij} \eta_{kl} \eta_{mn} + \dots, \end{aligned} \quad (12)$$

$$\begin{aligned} \rho_a F(\eta, T) &= \rho_a F(0, T) + \left(\frac{1}{2} \right) c_{ijkl}^T \eta_{ij} \eta_{kl} \\ &+ \left(\frac{1}{6} \right) c_{ijklmn}^T \eta_{ij} \eta_{kl} \eta_{mn} + \dots, \end{aligned} \quad (13)$$

where the adiabatic and isothermal second order elastic (SOE) constants at the natural state are respectively defined as

$$\begin{aligned} c_{ijkl}^S &= \left[\rho_a \frac{\partial^2 U(\eta, S)}{\partial \eta_{ij} \partial \eta_{kl}} \right]_{\mathbf{a}} = \left(\frac{\partial \tau_{ij}^a}{\partial \eta_{kl}} \right)_{S;\mathbf{a}}, \\ c_{ijkl}^T &= \left[\rho_a \frac{\partial^2 F(\eta, T)}{\partial \eta_{ij} \partial \eta_{kl}} \right]_{\mathbf{a}} = \left(\frac{\partial \tau_{ij}^a}{\partial \eta_{kl}} \right)_{T;\mathbf{a}}, \end{aligned} \quad (14)$$

and similarly, the adiabatic and isothermal third order elastic (TOE) constants at the natural state are defined by

$$\begin{aligned} c_{ijklmn}^S &= \left[\rho_a \frac{\partial^3 U(\eta, S)}{\partial \eta_{ij} \partial \eta_{kl} \partial \eta_{mn}} \right]_{\mathbf{a}} = \left(\frac{\partial^2 \tau_{ij}^a}{\partial \eta_{kl} \partial \eta_{mn}} \right)_{S;\mathbf{a}} \\ c_{ijklmn}^T &= \left[\rho_a \frac{\partial^3 F(\eta, T)}{\partial \eta_{ij} \partial \eta_{kl} \partial \eta_{mn}} \right]_{\mathbf{a}} = \left(\frac{\partial^2 \tau_{ij}^a}{\partial \eta_{kl} \partial \eta_{mn}} \right)_{T;\mathbf{a}}. \end{aligned} \quad (15)$$

It suffices to say that the internal and free energies in Equations 12 and 13 are truncated after the third order term in Lagrangian strain. The effective thermodynamic elastic stiffness coefficients determined from

precision ultrasonic measurements in the range below the elastic limit of a material indicates that they vary linearly with the Lagrangian strain, which can be predicted by truncating after the third order term. The fourth and higher order terms contribute to a nonlinear behavior of the elastic coefficients with strain. This nonlinearity is extremely small, very difficult to detect, and usually buried in experimental errors.

The thermodynamic elastic stiffness coefficients appearing in Equation 11, which are referred to and evaluated at the initial state, is related to the corresponding those, $c_{pqrs}^{T \text{ or } S}(\mathbf{X})$, referred to the natural state and evaluated at the initial state by [12, 13]

$$\begin{aligned} c_{ijkl}^{T \text{ or } S}(\mathbf{X}) &= \frac{\rho_X}{\rho_a} \frac{\partial X_i}{\partial a_p} \frac{\partial X_j}{\partial a_q} \frac{\partial X_k}{\partial a_r} \frac{\partial X_l}{\partial a_s} c_{pqrs}^{T \text{ or } S}(\mathbf{X}) \\ &= \frac{1}{\lambda_1^2 \lambda_3} \lambda_i \lambda_j \lambda_k \lambda_l c_{ijkl}^{T \text{ or } S}(\mathbf{X}), \end{aligned} \quad (16)$$

(i, j, k, l not summed)

where the λ 's with subscripts are the principal stretches defined in Equation 5 and

$$\begin{aligned} c_{pqrs}^{T \text{ or } S}(\mathbf{X}) &= \left(\rho_a \frac{\partial^2 F}{\partial \eta_{pq} \partial \eta_{rs}} \right)_{\mathbf{X}} \quad \text{or} \quad \left(\rho_a \frac{\partial^2 U}{\partial \eta_{pq} \partial \eta_{rs}} \right)_{\mathbf{X}} \\ &= c_{pqrs}^{T \text{ or } S}(\mathbf{a}) + c_{pqrsuv}^{T \text{ or } S}(\mathbf{a}) \eta_{uv} + \dots \end{aligned} \quad (17)$$

The SOE and TOE constants on the second line of Equation 17 are defined in Equations 14 and 15 respectively.

It is henceforth understood that when the superscripts T and S do not appear in the elastic coefficients, such as c_{ijkl} , c_{ijklmn} , C_{ijkl} , the Young's modulus and Poisson's ratio, these elastic coefficients are referred to either isothermal or adiabatic conditions. In most cases in this work the strains are defined with reference to stress-free natural state. So, when there appears no superscript as in the thermodynamic stress τ_{ij} , it is also understood that $\tau_{ij} = \tau_{ij}^a$. It follows from Equations 7, 10, 12 and 13 that for the homogeneous uniaxial loading in the X_3 direction

$$\sigma_{ij}(\mathbf{X}) = \frac{1}{\lambda_1^2 \lambda_3} \lambda_i \lambda_j \left(c_{ijkl} \eta_{kl} + \frac{1}{2} c_{ijklmn} \eta_{kl} \eta_{mn} + \dots \right) \quad (i, j \text{ not summed}). \quad (18)$$

Since the strains, the thermodynamic and Cauchy stresses are symmetric with respect to the subscript indices, it is convenient to introduce the Voigt notation: 11 ~ 1, 22 ~ 2, 33 ~ 3, 23 ~ 4, 13 ~ 5, 12 ~ 6. Using the Voigt notation, a material of cubic symmetry has three SOE constants:

$$c_{11} = c_{22} = c_{33}, \quad c_{12} = c_{13} = c_{23}, \quad c_{44} = c_{55} = c_{66}, \quad (19)$$

and six TOE constants (432, $\bar{4}3m$, $m\bar{3}m$ groups):

$$\begin{aligned} c_{111} = c_{222} = c_{333}, \quad c_{144} = c_{255} = c_{366}, \quad c_{123}, \quad c_{456}, \\ c_{112} = c_{223} = c_{133} = c_{113} = c_{122} = c_{233}, \quad (20) \\ c_{155} = c_{244} = c_{344} = c_{166} = c_{266} = c_{355}. \end{aligned}$$

For an isotropic material there are two SOE constants:

$$c_{12} = \lambda, \quad c_{44} = \mu, \quad c_{11} = \lambda + 2\mu, \quad (21)$$

and three TOE constants:

$$\begin{aligned} c_{123} &= \nu_1, \quad c_{144} = \nu_2, \quad c_{456} = \nu_3, \\ c_{112} &= \nu_1 + 2\nu_2, \quad c_{155} = \nu_2 + 2\nu_3, \\ c_{111} &= \nu_1 + 6\nu_2 + 8\nu_3, \end{aligned} \quad (22)$$

where λ and μ are called the second order Lamé constants, and ν_1 , ν_2 , and ν_3 are called the third order Lamé constants.

For notational simplicity and convenience for the equations we deal with in the following sections, let us first define dimensionless parameters, ν_0 , h_1 , h_2 , h_3 , g_1 , g_2 , b_1 , and b_2 , as

$$\nu_0 \equiv \frac{c_{12}}{c_{11} + c_{12}}, \quad (23)$$

$$\begin{aligned} h_1 &\equiv \frac{c_{112}}{2c_{12}}, \quad h_2 \equiv \frac{c_{112} + c_{123}}{c_{11} + c_{12}} = \frac{\nu_0(c_{112} + c_{123})}{c_{12}}, \\ h_3 &\equiv \frac{c_{111} + 3c_{112}}{2(c_{11} + c_{12})}, \end{aligned} \quad (24)$$

$$g_1 \equiv h_1 - h_2 + \nu_0 h_3,$$

$$g_2 \equiv h_2^2 - h_1 h_2 + \nu_0 h_3 (2h_1 - 3h_2) + 2\nu_0^2 h_3^2, \quad (25)$$

$$b_1 \equiv 2(1 + \nu_0 + g_1), \quad (26)$$

$$b_2 \equiv 4\nu_0 + 4\nu_0^2 + 6\nu_0 g_1 + 4g_1 + 3g_2,$$

and next define strength parameters, E_0 , e_1 , and e_2 , as

$$E_0 \equiv c_{11} - 2\nu_0 c_{12}, \quad (27)$$

$$e_1 \equiv E_0(1 + 2\nu_0) + c_{12}(\nu_0^{-1} h_3 - 3h_1 - 6\nu_0 h_1 + 3\nu_0 h_2 - 2\nu_0^2 h_3),$$

$$\begin{aligned} e_2 &\equiv E_0(2\nu_0 + 4\nu_0^2 + 2\nu_0 g_1 - 0.5) \\ &+ c_{12}[(\nu_0^{-1} + 2)(h_3 - 3\nu_0 h_1 - 6\nu_0^2 h_1 \\ &+ 3\nu_0^2 h_2 - 2\nu_0^3 h_3) - 2\nu_0(2h_1 g_1 - h_2 g_1 + g_2)]. \end{aligned} \quad (28)$$

Note that E_0 in Equation 27 and ν_0 in Equation 23 are the Young's modulus and Poisson's ratio of a cubic material at the stress-free natural state.

A cubic crystal loaded in the [001] direction tends towards that of tetragonal symmetry and may be termed as having a tetratropic symmetry, which has six thermodynamic elastic stiffness coefficients of the second order defined in Equation 11. They are written as

$$\begin{aligned} C_{11} &= C_{22}, \quad C_{33}, \quad C_{12}, \quad C_{13} = C_{23}, \\ C_{44} &= C_{55}, \quad C_{66}. \end{aligned} \quad (29)$$

Formulas for phase and group velocities in terms of $C_{\mu\nu}$ and stresses acting on the medium are described in detail for the symmetry planes of stressed anisotropic solids including a tetratropic medium by Kim *et al.* [24]

3. Poisson's ratio and volume change

In the aforementioned homogenous loading in the X_3 direction coinciding with the [001] cubic axis, the only nonzero stress component is $\sigma_{33} = \sigma_3 \neq 0$. The nonzero Lagrangian strains are three principal strains, which are $\eta_{ij} = \eta_i \delta_{ij}$ with $\eta_1 = \eta_2$. Using Equation 18 with the identity relations given in Equations 19 and 20, one obtains for the stress $\sigma_1 = \sigma_{11} = 0$

$$\begin{aligned} \sigma_1 &= \frac{1}{\lambda_3} \left(c_{11kl} \eta_k \delta_{kl} + \frac{1}{2} c_{11klmn} \eta_k \delta_{kl} \eta_m \delta_{mn} + \dots \right) \\ &= \frac{1}{\lambda_3} \left[(c_{11} + c_{12}) \eta_1 + c_{12} \eta_3 + \frac{1}{2} \{ (c_{111} + 3c_{112}) \eta_1^2 \right. \\ &\quad \left. + 2(c_{112} + c_{123}) \eta_1 \eta_3 + c_{112} \eta_3^2 \} + \dots \right] = 0. \end{aligned} \quad (30)$$

Let

$$k \equiv -\frac{\eta_1}{\eta_3}, \quad \text{i.e.,} \quad \eta_1 \equiv -k \eta_3. \quad (31)$$

Substituting η_1 by $-k \eta_3$ into Equation 30 and solving the resulting quadratic equation in k in terms of η_3 yield

$$k = \frac{1 + h_2 \eta_3}{2h_3 \eta_3} \left[1 - \left\{ 1 - \frac{4\nu_0 h_3 \eta_3 (1 + h_1 \eta_3)}{(1 + h_2 \eta_3)^2} \right\}^{1/2} \right], \quad (32)$$

where ν_0 , the Poisson's ratio at the stress-free state, and h_1 , h_2 , and h_3 are given by Equations 23 and 24, respectively. We have chosen $k > 0$ for a normal solid. The solution with a positive square root for k corresponds to a negative k and is discarded in Equation 32.

Using the binomial theorem, Equation 32 is expanded in powers of η_3 to yield after an involved algebra

$$k = \nu_0(1 + g_1 \eta_3 + g_2 \eta_3^2 + g_3 \eta_3^3 + \dots), \quad (33)$$

where g_1 and g_2 are given in Equation 25 and

$$\begin{aligned} g_3 &\equiv h_2^2(h_1 - h_2) + \nu_0 h_3(h_1^2 - 6h_1 h_2 + 6h_2^2) \\ &+ 2\nu_0^2 h_3^2(3h_1 - 5h_2). \end{aligned} \quad (34)$$

The effective Poisson's ratio at the initial state \mathbf{X} is independent of direction in the $X_1 X_2$ plane normal to the loading direction and we denote it by ν_3 , which is defined by

$$\nu_3 \equiv -\frac{d\lambda_1/\lambda_1}{d\lambda_3/\lambda_3} = -\frac{1 + 2\eta_3}{1 + 2\eta_1} \frac{d\eta_1}{d\eta_3}. \quad (35)$$

$$-\frac{d\eta_1}{d\eta_3} = \frac{d[k(\eta_3)\eta_3]}{d\eta_3} = \nu_0(1 + 2g_1 \eta_3 + 3g_2 \eta_3^2 + \dots), \quad (36)$$

which is substituted into Equation 35 to yield

$$\nu_3(\mathbf{X}) = \nu_0(1 + b_1 \eta_3 + b_2 \eta_3^2 + \dots), \quad (37)$$

where b_1 and b_2 are expressed in Equation 26. The effective Poisson's ratio is a ratio of an infinitesimal transverse strain to an infinitesimal longitudinal strain of a material referred to the very strained state. Hence,

when we say simply the Poisson's ratio, it actually means the effective Poisson's ratio. An engineering strain ε_3 is related to a Lagrangian strain η_3 via Equation 3 by

$$\eta_3 = \varepsilon_3 \left(1 + \frac{1}{2} \varepsilon_3 \right). \quad (38)$$

The Poisson's ratio can be now expressed in terms of engineering strain as

$$\nu_3 = \nu_0 \left[1 + b_1 \varepsilon_3 + \left(\frac{1}{2} b_1 + b_2 \right) \varepsilon_3^2 + \dots \right]. \quad (39)$$

The engineering Poisson's ratio ν_3^E is defined as

$$\nu_3^E \equiv -\frac{\varepsilon_1}{\varepsilon_3}. \quad (40)$$

In terms of the engineering strain and engineering Poisson's ratio, k in Equation 31 is expressed as

$$k \equiv -\frac{\eta_1}{\eta_3} = -\frac{\varepsilon_1(1 + \varepsilon_1/2)}{\varepsilon_3(1 + \varepsilon_3/2)} = \frac{\nu_3^E(1 - \nu_3^E \varepsilon_3/2)}{(1 + \varepsilon_3/2)}, \quad (41)$$

which in combination with Equations 33 and 38 yields a quadratic equation in ν_3^E . Solving for ν_3^E in terms of ε_3 and opting for $\nu_3^E > 0$ for a normal solid, one expresses a strain dependence of the engineering Poisson's ratio as

$$\nu_3^E = \nu_0 \left[1 + \frac{1}{2} (1 + \nu_0 + 2g_1) \varepsilon_3 + \frac{1}{2} (\nu_0 + \nu_0^2 + 2\nu_0 g_1 + 2g_1 + 2g_2) \varepsilon_3^2 + \dots \right]. \quad (42)$$

Note that in a usual tension or compression testing of materials, what are conventionally measured are an engineering strain and an applied force, and the engineering Poisson's ratio, which is treated as a constant. Equations 39 and 42 indicate, however, that both the (effective) Poisson's ratio and engineering Poisson's ratio are a function of strain. These strain dependencies are amply demonstrated by precision ultrasonic measurements of elastic moduli [15, 16]. What has been so far unknown are the explicit analytical expressions, such as Equations 37, 39 and 42.

It may be interesting to see if the Poisson's ratio $\nu_3(\mathbf{X})$ given in Equations 87 and 88 of Ref. [14], where it is expressed in terms of the second order thermodynamic elastic stiffness coefficients given by Equation 11 and three principal stresses, will lead to the identical results for strain dependence. Applying Equation 87 of Ref. [14] to the case of uniaxial homogenous loading of $\sigma_1 = \sigma_2 = 0$, one obtains, with the aid of Equations 16, 17, and 29, an expression

$$\nu_3(\mathbf{X}) \equiv -\frac{d\lambda_1/\lambda_1}{d\lambda_3/\lambda_3} = \frac{C_{13}}{C_{11} + C_{12}} = \frac{\lambda_3^2}{\lambda_1^2} \times \frac{c_{12} + (c_{112} + c_{123})\eta_1 + c_{112}\eta_3}{c_{11} + c_{12} + (c_{111} + 3c_{112})\eta_1 + (c_{112} + c_{123})\eta_3}, \quad (43)$$

which yields the following differential equation involving η_1 and η_3 :

$$-\frac{d\eta_1}{d\eta_3} = \frac{\nu_0 + h_2\eta_1 + 2h_1\nu_0\eta_3}{1 + 2h_3\eta_1 + h_2h_3}. \quad (44)$$

The solution of the above differential equation leads to $k \equiv -\eta_1/\eta_3$ being identical to Equation 32.

A change in cross-sectional area, A_X/A_a , is specified by λ_1^2 , which can be expressed as

$$\lambda_1^2 = A_X/A_a = 1 + 2\eta_1 = 1 - 2k\eta_3 = 1 - 2\nu_0\eta_3(1 + g_1\eta_3 + g_2\eta_3^2 + \dots). \quad (45)$$

A change in volume V or the density ρ from the natural state to the initial state is given by

$$\frac{\rho_a}{\rho_X} = \frac{V_X}{V_a} = \lambda_1^2 \lambda_3 = 1 + (1 - 2\nu_0)\eta_3 - (0.5 + 2\nu_0 + 2\nu_0 g_1)\eta_3^2 + \dots = 1 + (1 - 2\nu_0)\varepsilon_3 - \nu_0(3 + 2g_1)\varepsilon_3^2 + \dots. \quad (46)$$

4. Nonlinear elastic equation of state and Young's modulus

It follows from Equations 18, 19 and 20 that the only nonzero stress component σ_3 is expressed as

$$\begin{aligned} \sigma_3 &= \frac{\lambda_3}{\lambda_1^2} \tau_{33} \\ &= \frac{\lambda_3}{\lambda_1^2} \left(c_{33ij} \eta_i \delta_{ij} + \frac{1}{2} c_{33ijkl} \eta_i \delta_{ij} \eta_k \delta_{kl} + \dots \right) \\ &= \frac{(1 + 2\eta_3)^{1/2}}{1 + 2\eta_1} \left[2c_{12}\eta_1 + c_{11}\eta_3 + \left\{ (c_{112} + c_{123})\eta_1^2 + 2c_{112}\eta_1\eta_3 + \frac{1}{2}c_{111}\eta_3^2 \right\} + \dots \right]. \end{aligned} \quad (47)$$

Substituting η_1 by $-k\eta_3$, where k is given by Equation 32, and expanding the resulting equation in power series of η_3 , one obtains after some lengthy algebra

$$\sigma_3 = \eta_3 (E_0 + e_1\eta_3 + e_2\eta_3^2 + \dots), \quad (48)$$

where E_0 , the Young's modulus at the relaxed state is given in Equation 27, and e_1 and e_2 are defined by Equation 28. The (Cauchy) stress at the initial state, σ_3 , is related to an engineering strain via Equation 38 by

$$\sigma_3 = \varepsilon_3 \left[E_0 + \left(\frac{1}{2} E_0 + e_1 \right) \varepsilon_3 + (e_1 + e_2) \varepsilon_3^2 + \dots \right]. \quad (49)$$

The first Piola-Kirchhoff stress referred to the original area at the natural state is known as an engineering stress σ_3^E in a conventional tension testing of materials. Using Equations 45 and 38, the engineering stress can be expressed in terms of engineering strain as

$$\sigma_3^E = \lambda_1^2 \sigma_3 = \varepsilon_3 \left[E_0 + \left(\frac{1}{2} E_0 - 2\nu_0 E_0 + e_1 \right) \varepsilon_3 - (2\nu_0 E_0 + 2\nu_0 g_1 E_0 - e_1 + 2\nu_0 e_1 - e_2) \varepsilon_3^2 + \dots \right]. \quad (50)$$

Note that in the conventional tension (or compression) testing of materials, σ_3^E versus ε_3 are usually plotted with a linear dependence between them. However, the above equation implies that the relation is basically nonlinear.

With the knowledge of Equation 48, the (effective) Young's modulus E_3 at the initial state \mathbf{X} can be written as

$$\begin{aligned} E_3(\mathbf{X}) &\equiv \frac{d\sigma_3}{d\lambda_3/\lambda_3} = \lambda_3 \frac{d\sigma_3}{d\eta_3} \frac{d\eta_3}{d\lambda_3} \\ &= \lambda_3^2 (E_0 + 2e_1 \eta_3 + 3e_2 \eta_3^2 + \dots) \\ &= E_0 + 2(E_0 + e_1) \eta_3 + (4e_1 + 3e_2) \eta_3^2 + \dots, \end{aligned} \quad (51)$$

which is expressed in terms of engineering strain as

$$E_3 = E_0 + 2(E_0 + e_1) \varepsilon_3 + (E_0 + 5e_1 + 3e_2) \varepsilon_3^2 + \dots. \quad (52)$$

Equation 50 readily yields an expression for the engineering Young's modulus E_3^E defined as σ_3^E/ε_3 :

$$\begin{aligned} E_3^E &\equiv \frac{\sigma_3^E}{\varepsilon_3} = E_0 + \left(\frac{1}{2} E_0 - 2\nu_0 E_0 + e_1 \right) \varepsilon_3 \\ &\quad - (2\nu_0 E_0 + 2\nu_0 g_1 E_0 - e_1 + 2\nu_0 e_1 - e_2) \varepsilon_3^2 + \dots. \end{aligned} \quad (53)$$

The engineering Young's modulus can be construed as a kind of average of the Young's moduli between the natural and initial states. Although its physical meaning is not so clear-cut, it is the most easily and conveniently determined modulus in a conventional testing of materials. The linear infinitesimal elastic theory treats it as a constant, while the finite deformation theory predicts a strain dependence of the Young's modulus. Equations 48–53 are all considered as nonlinear elastic equations of state of solids, which describe a nonlinear elastic relation between the stress and the strain of materials.

Referring to Equation 89 of Ref. [14], one writes the (effective) Young's modulus in the uniaxial loading along the X_3 direction as

$$E_3(\mathbf{X}) = C_{33} + \sigma_3 - 2\nu_3(C_{13} - \sigma_3), \quad (54)$$

where C_{33} and C_{13} are the thermodynamic elastic stiffness coefficients of the second order defined in Equation 11 and the (effective) Poisson's ratio ν_3 is given by Equation 39. Expressing the C_{33} and C_{13} in terms of the second and third order elastic constants via Equations 16 and 17 and substituting the result, Equations 37 and 48 into Equation 54, one can prove after a lengthy involved algebra that Equation 54 is identical to Equation 51, providing consistency to the results we obtained.

5. Stress dependence of strain, Young's modulus, Poisson's ratio and volume change

So far, we have expressed various elastic moduli and the Poisson's ratios as a function of strain. In this section we will express them as an explicit function of stress. To obtain the stress dependence of the Young's modulus and Poisson's ratio, Equations 48–50 are inverted to express a strain as a function of stress. Expressing the results in powers of σ_3/E_0 or σ_3^E/E_0 , one obtains

$$\eta_3 = \frac{\sigma_3}{E_0} \left(1 - \frac{e_1}{E_0} \frac{\sigma_3}{E_0} + \left(\frac{2e_1^2}{E_0^2} - \frac{e_2}{E_0} \right) \left(\frac{\sigma_3}{E_0} \right)^2 \dots \right), \quad (55)$$

$$\begin{aligned} \varepsilon_3 &= \frac{\sigma_3}{E_0} \left[1 - \left(\frac{1}{2} + \frac{e_1}{E_0} \right) \frac{\sigma_3}{E_0} + \left(\frac{1}{2} + \frac{e_1}{E_0} + 2 \frac{e_1^2}{E_0^2} \right. \right. \\ &\quad \left. \left. - \frac{e_2}{E_0} \right) \left(\frac{\sigma_3}{E_0} \right)^2 \dots \right], \end{aligned} \quad (56)$$

$$\begin{aligned} \varepsilon_3 &= \frac{\sigma_3^E}{E_0} \left[1 + \left(2\nu_0 - \frac{1}{2} - \frac{e_1}{E_0} \right) \frac{\sigma_3^E}{E_0} \right. \\ &\quad \left. + \left(\frac{1}{2} - 2\nu_0 + 2\nu_0 g_1 + 8\nu_0^2 \right. \right. \\ &\quad \left. \left. + \frac{e_1}{E_0} - 6\nu_0 \frac{e_1}{E_0} + \frac{2e_1^2}{E_0^2} - \frac{e_2}{E_0} \right) \left(\frac{\sigma_3^E}{E_0} \right)^2 \dots \right]. \end{aligned} \quad (57)$$

Substituting Equations 56 and 57 into Equations 39, 42, 46, 52, and 53, the variations of the Poisson's ratio, Young's modulus, and a volume change with stress can be obtained. We express the effective Poisson's ratio and effective Young's modulus as a function of Cauchy stress, the engineering Poisson's ratio and engineering Young's modulus in terms of engineering stress, and a volume change using both Cauchy and engineering stresses. They are written as

$$\nu_3 = \nu_0 \left[1 + b_1 \frac{\sigma_3}{E_0} + \left(b_2 - \frac{b_1 e_1}{E_0} \right) \left(\frac{\sigma_3}{E_0} \right)^2 + \dots \right], \quad (58)$$

$$\begin{aligned} E_3 &= E_0 + 2(E_0 + e_1) \frac{\sigma_3}{E_0} \\ &\quad + \left(2e_1 - 2 \frac{e_1^2}{E_0} + 3e_2 \right) \left(\frac{\sigma_3}{E_0} \right)^2 + \dots, \end{aligned} \quad (59)$$

$$\begin{aligned} \frac{V_X}{V_a} &= \frac{\rho_a}{\rho_X} = 1 + (1 - 2\nu_0) \frac{\sigma_3}{E_0} \\ &\quad - \left(\frac{1}{2} + 2\nu_0 + 2\nu_0 g_1 + \frac{e_1}{E_0} - \frac{2\nu_0 e_1}{E_0} \right) \\ &\quad \times \left(\frac{\sigma_3}{E_0} \right)^2 + \dots, \end{aligned} \quad (60)$$

$$\begin{aligned} &= 1 + (1 - 2\nu_0) \frac{\sigma_3^E}{E_0} - \left(\frac{1}{2} + 4\nu_0^2 + 2\nu_0 g_1 + \frac{e_1}{E_0} \right. \\ &\quad \left. - \frac{2\nu_0 e_1}{E_0} \right) \left(\frac{\sigma_3^E}{E_0} \right)^2 + \dots \end{aligned}$$

$$\nu_3^E = \nu_0 \left[1 + \frac{1}{2}(1 + \nu_0 + 2g_1) \frac{\sigma_3^E}{E_0} + \frac{1}{2} \left(\frac{5}{2} \nu_0 + 3\nu_0^2 - \frac{1}{2} + g_1 + 6\nu_0 g_1 + 2g_2 - \frac{e_1}{E_0} - \frac{\nu_0 e_1}{E_0} - \frac{2g_1 e_1}{E_0} \right) \left(\frac{\sigma_3^E}{E_0} \right)^2 \dots \right], \quad (61)$$

$$E_3^E = E_0 + \left(\frac{1}{2} E_0 - 2\nu_0 E_0 + e_1 \right) \frac{\sigma_3^E}{E_0} - \left(\frac{1}{4} E_0 + 4\nu_0^2 E_0 + 2\nu_0 g_1 E_0 - 2\nu_0 e_1 + \frac{e_1^2}{E_0} - e_2 \right) \left(\frac{\sigma_3^E}{E_0} \right)^2 \dots \quad (62)$$

Equations 57 and 62, in which a stress σ_3^E is defined with respect to unit area at the stress-free state, are derived in Refs. 2 and 7 without the last term in both equations.

6. Illustrations of nonlinear elastic behavior

It is well known in ultrasonics that sound speeds vary with applied stresses. From measurements of the variation of the sound speeds with uniaxial stress and hydrostatic pressures, one can determine the SOE and TOE constants. A method of determining the (effective) Young's modulus and Poisson's ratio at an arbitrarily stressed initial state from the ultrasonic acoustoelastic measurements has been described by the authors [25]. Variations of the Young's modulus and Poisson's ratio with stress or strain have been observed to be linear in most materials in the stress range below the elastic limit. The square terms in stress and strain, which are shown in Equations 39, 52, 58 and 59, are so small that they are difficult to detect experimentally. These equations are derived on the assumption that the internal and free energies truncated after the third order term in Lagrangian strain (see Equations 12 and 13) correctly describe the material behavior. Even though the assumption fits the material behavior very well, it is not strictly true, and the internal and free energies should in principle include the fourth and higher order terms in Lagrangian strain. In such a case the cubic terms in strain (or stress) in the expressions of the constitutive elastic equations of state, such as Equations 49, 50, 56, and 57, and the square terms in strain (or stress) in the expressions of the Young's modulus and Poisson's ratio, such as Equations 39, 42, 52, 53, 58, 59, 61, and 62, will be affected by the inclusion of the fourth order elastic (FOE) constants. The FOE constants of some materials can be theoretically estimated (see Ref. [11]). However, they cannot be reliably determined by the present status of art in experiment. In the following examples taken with cubic silicon and isotropic carbon steel, Pyrex glass, and polystyrene, we neglect these cubic and square terms. They are retained in the equations given in Sections 3–5, as they represent a behavior of higher order nonlinearity in case that the FOE constants are nearly zero and an observed elastic behavior of materials exhibits such a higher order nonlinearity.

The SOE and TOE constants of cubic silicon are taken from the measurement by Hall [26], who reported in units of GPa $c_{11} = 165.64$, $c_{12} = 63.94$, $c_{44} = 79.51$, $c_{111} = -795$, $c_{112} = -445$, $c_{123} = -75$, $c_{144} = 15$, $c_{155} = -310$, $c_{456} = -86$. Using these constants, we calculate the relevant parameters in Equations 23–28 to express the nonlinear elastic equation of state and write the (effective) Young's modulus, Poisson's ratio, and a volume change as a function of Cauchy stress and engineering strain.

$$\sigma_3 = \varepsilon_3(130.02 + 166.78\varepsilon_3 + \dots), \quad (63)$$

$$E_3 = 130.02 + 3.565\sigma_3 + \dots = 130.02 + 463.59\varepsilon_3 + \dots, \quad (64)$$

$$\nu_3 = 0.2785(1 - 0.01889\sigma_3 + \dots) = 0.2785(1 - 2.457\varepsilon_3 + \dots), \quad (65)$$

$$\frac{V_X}{V_a} = \frac{\rho_a}{\rho_X} = 1 + 0.003407\sigma_3 + 4.382 \times 10^{-7}\sigma_3^2 + \dots = 1 + 0.4430\varepsilon_3 + 0.5608\varepsilon_3^2 + \dots \quad (66)$$

Next, we turn to a typical structural isotropic material, high-strength carbon steel Hecla 37 (0.4% C, 0.3% Si, 0.8% Mn), the SOE and TOE Lamé constants of which were reported in units of GPa by Smith *et al.* [27] as $\lambda = 111$, $\mu = 82.1$, $\nu_1 = -358$, $\nu_2 = -282$, and $\nu_3 = -177$. We choose to express a nonlinear elastic equation of state that relates an engineering stress to an engineering strain and to list other quantities as a function of engineering strain, as they are typically measured in a conventional tension/compression testing. They are written as

$$\sigma_3^E = \varepsilon_3(211.39 - 790.2\varepsilon_3 + \dots), \quad (67)$$

where the term inside the parenthesis is equal to the engineering Young's modulus E_3^E , and

$$\nu_3^E = 0.2874(1 - 2.514\varepsilon_3 + \dots), \quad (68)$$

$$\frac{V_X}{V_a} = \frac{\rho_a}{\rho_X} = 1 + 0.42517\varepsilon_3 + 0.9529\varepsilon_3^2 + \dots \quad (69)$$

Now, we choose Pyrex glass, an amorphous material to see its nonlinear elastic behavior, whose SOE and TOE Lamé constants were reported [16] in units of GPa as $\lambda = 13.53$, $\mu = 27.5$, $\nu_1 = 264$, $\nu_2 = -118$, and $\nu_3 = 105$. Pyrex glass has an anomalous behavior in the TOE constants in the sense that ν_1 and ν_3 are positive, in lieu of being negative just as those of other typical isotropic materials are. This behavior is phenomenologically associated with the low thermal expansion coefficient of Pyrex glass. A similar behavior is found in fused quartz [28], which has a very low thermal expansion coefficient.

$$\sigma_3 = \varepsilon_3(64.07 + 323.0\varepsilon_3 \dots), \quad (70)$$

$$\sigma_3^E = \varepsilon_3(64.07 + 301.9\varepsilon_3 \dots), \quad (71)$$

$$E_3 = 64.07 + 11.08\sigma_3 \dots = 64.07 + 710.2\varepsilon_3 \dots, \quad (72)$$

$$\begin{aligned} \nu_3 &= 0.1649(1 - 0.02736\sigma_3 \dots) \\ &= 0.1649(1 - 1.753\varepsilon_3 \dots), \end{aligned} \quad (73)$$

$$\begin{aligned} \frac{V_X}{V_a} &= 1 + 0.01046\sigma_3 + 7.798 \times 10^{-4}\sigma_3^2 \dots \\ &= 1 + 0.6702\varepsilon_3 + 0.1785\varepsilon_3^2 \dots \end{aligned} \quad (74)$$

Equations 70–73 indicate that Pyrex glass, a brittle material of medium strength, has a strong nonlinear contribution to its elastic equation of state and this kind of strong dependence of the Young's modulus and Poisson's ratio on stress (or strain) may be typical of a behavior of glass that possesses a low thermal expansion coefficient.

Finally, we take an example of a typical polymeric material, polystyrene, the SOE and TOE Lamé constants of which were reported by Hughes and Kelly [16] to be in units of GPa $\lambda = 2.889$, $\mu = 1.381$, $\nu_1 = -21.2$, $\nu_2 = -8.3$, and $\nu_3 = -2.5$. An engineering strain, the engineering Young's modulus, engineering Poisson's ratio, and a volume change, are expressed in terms of engineering stress. The nonlinear elastic equations of state, the Young's modulus, Poisson's ratio, and a change in volume/density are written as

$$\varepsilon_3 = \sigma_3^E (0.2705 + 0.2759\sigma_3^E \dots), \quad (75)$$

where the quantity inside the parenthesis is equal to a reciprocal of the engineering Young's modulus E_3^E ,

$$\nu_3^E = 0.3383(1 - 0.8150\sigma_3^E \dots), \quad (76)$$

$$\frac{V_X}{V_a} = 1 + 0.08750\sigma_3^E + 0.1973(\sigma_3^E)^2 \dots \quad (77)$$

Equations 75 and 76 indicate that polystyrene has a relatively high nonlinear contribution to the elastic equation of state. A strong dependence of the Young's modulus and Poisson's ratio on stress (or strain) is shown by them. This kind of behavior might have been expected for low-strength materials such as polymers. However, a volume change is quite small even at large strain $\varepsilon_3 = 0.1$.

Note that in the above Equations 63–77 the highest order terms of strain and stress, which are absent in the conventional linear theory, are due to the contributions of the third order terms in the internal and free energies given in Equations 12 and 13.

7. Discussion

A conventional tension/compression testing of materials lacks in precision measurements of strain and usually fails to show a nonlinear behavior between stress and strain, which is described in Sections 4 and 5. In an ordinary tension/compression test, the calculation of the Young's modulus involves the differentiation of stress with respect to strain, which again involves the differentiation of displacement with respect to gauge dimension, and therefore, a determination of the Young's modulus accurate enough to exhibit a strain dependence requires very precise measurements of the applied force

and dimensional changes of a specimen. Better accuracy in the (effective) Young's modulus and often in the Poisson's ratio is usually obtained through accurate measurements of ultrasonic wavespeeds of various modes propagating along various directions in the specimen, as described in Ref. [25]. Since the precision ultrasonic measurements, which determine sound speeds better than one part in 10^4 , exhibit a nonlinear behavior between stress and strain, it should be possible to see the nonlinear behavior in a carefully controlled tension/compression testing which provides precision measurements of an applied load and dimensions of a specimen. Indeed, a nonlinear stress-strain relation and a strain dependence of the Young's modulus were demonstrated in a tension testing of various whiskers [29, 30]. They can be used to determine some of the TOE constants of materials.

The Young's modulus and Poisson's ratio obtained from the tension/compression test are isothermal values, while those calculated from the ultrasonic measurements via Equations 43 and 54 are adiabatic ones. Denoting the isothermal and adiabatic values by superscripts T and S , respectively, they are related by [14]

$$E_3^T = \frac{E_3^S}{1 + \frac{\alpha_3^\sigma \alpha_3^\tau E_3^S T}{\rho_X C_\sigma} (1 + r_t)} = \frac{E_3^S}{1 + \alpha_3^\sigma \lambda E_3^S}, \quad (79)$$

$$E_3^S = \frac{E_3^T}{1 - \alpha_3^\sigma \lambda E_3^T},$$

$$\nu_3^S = \frac{\nu_3^T + \alpha_1^\sigma \lambda E_3^T}{1 - \alpha_3^\sigma \lambda E_3^T}, \quad (80)$$

where λ denotes

$$\lambda \equiv \frac{T \alpha_3^\tau (1 + r_t)}{\rho_X C_\sigma}, \quad (81)$$

and the dimensionless parameter r_t is expressed as

$$r_t = \left(\frac{\alpha_1^\tau}{\alpha_3^\tau} \right) Q_{33}^T \sigma_1 + \left(\frac{\alpha_2^\tau}{\alpha_3^\tau} \right) Q_{33}^T \sigma_2 + (2Q_{13}^T - Q_{33}^T) \sigma_3. \quad (82)$$

In Equations 79–82, α_i^σ and α_i^τ are the thermal expansion coefficients at constant Cauchy stress and constant thermodynamic stress, respectively, C_σ is the specific heat at constant Cauchy stress, and Q_{ij}^T are the effective isothermal elastic compliance coefficients. For detail of these thermodynamic relations, refer to Ref. [14]. The terms $\alpha_\mu^\sigma \lambda E_3^T$ or S ($\mu = 1, 2, 3$) in Equations 79 and 80 depend on stress. However, they are much smaller than unity and may be replaced by the constant values evaluated at the stress-free natural state. For conversion between the thermodynamic elastic stiffness coefficients $C_{\mu\nu}^T$ and $C_{\mu\nu}^S$, between the SOE constants $c_{\mu\nu}^T$ and $c_{\mu\nu}^S$, between the TOE constants $c_{\lambda\mu\nu}^T$ and $c_{\lambda\mu\nu}^S$ ($\lambda, \mu, \nu = 1, 2 \dots 6$), refer to Refs. [14] and [9].

Amorphous and nontextured polycrystalline materials possess isotropic symmetry at the stress-free natural

state and a majority of crystals observed in nature belong to cubic symmetry at the relaxed state. It is well known that an isotropic material at the stress-free natural state behaves as a transversely isotropic material at the stressed state under uniaxial loading, say in the X_3 direction. Its effective elastic moduli are characterized by five second-order thermodynamic elastic stiffness coefficients $C_{11} = C_{22}, C_{33}, C_{12}, C_{13} = C_{23}, C_{44} = C_{55}, C_{66} = (C_{11} - C_{12})/2$, which are related to the two SOE and three TOE Lamé constants via Equations 16, 17, 21 and 22. Similarly as aforementioned, a cubic material at the stress-free natural state behaves as a tetraotropic material at the stressed state under uniaxial loading in the X_3 direction. Its effective elastic moduli are characterized by six second-order thermodynamic elastic stiffness coefficients given by Equation 29, which are related to the three SOE and six TOE constants via Equations 16, 17, 19 and 20. Under triaxial stresses, an isotropic material at the natural state behaves as an orthotropic material with nine second-order thermodynamic elastic stiffness coefficients and so does a cubic material at the natural state when its cubic axes coincide with three principal stress directions. A similar procedure could in principle be extended to obtain nonlinear elastic relations for an orthotropic material. However, because of the increasing number of elastic constants involved and the presence of cross-coupling terms between principal stresses, one would expect very complicated nonlinear relations and it is hard to imagine that one might gain further meaningful physical insight from these relations.

8. Conclusion

Nonlinear elastic equations of states of solids governing the behavior between stress and strain under uniaxial homogeneous loading have been derived for cubic and isotropic solids. A strain/stress dependence of the Young's modulus and Poisson's ratio, and a change in volume/density is also expressed. These relations are described in terms of the SOE and TOE constants of solids. The nonlinear behaviors are illustrated with examples of cubic silicon, high-strength carbon steel, medium-strength Pyrex glass, and low-strength polystyrene.

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