Design and Construction of an Absolute Precision Load/Stress Gauge

Part I. Theoretical Methods

Kwang Yul Kim[a]

130 Fieldstone Cir, Ithaca, New York 14850

[a]) kyk1@cornell.edu

[a]) ORCID ID: 0000-0003-1338-5301

ABSTRACT

This paper, Part I of the article series "Design and Construction of an Absolute Precision Load/Stress Gauge," presents the formulation that calculates an applied uniaxial load/stress on an absolute precision load/stress gauge (APLSG) by using the finite deformation theory of an elastic solid, which was initially isotropic at a strain-free state; design and construction of the APLSG; and validity of the APLSG by testing it under a compression machine with a maximum capacity

[a]) Author's Website: aplgllc.org

of 300 imperial tons. The load-carrying member of the APLSG is a cylindrically shaped 7075 aluminum alloy of diameter 122.61mm. The vertically applied load is calculated using four measured data: a lateral dimensional change of a specimen in the horizontal direction, three travel times of horizontally propagating longitudinal waves, vertically polarized shear (SV) waves, and horizontally polarized shear (SH) waves. These data can be easily measured in experiments with great accuracy. The theory takes care of the linear and nonlinear elastic contributions of material behavior under finite deformation, contributing to great precision for the calculation of the applied load. The accuracy of the calculated load is better than 0.1%. The APLSG directly calculates the applied load in units of force or mass, and the applied stress in units of (mega) pascal, and thus precludes the need for its calibration, providing an advantage over the conventional load cells, which output the applied load in units of electrical units.

To the author's knowledge, this paper provides for the first time precise analytic relations between non-hydrostatic uniaxial stress and strain, contributing greatly to the equation of state of solids in materials science and applied physics.

Key Words: Precision Load Gauge; Precision Stress Gauge; Isothermal Equation of Solids; Finite Deformation; Thermodynamic Stress; Nonlinear Elastic Deformation

I. INTRODUCTION

In the finite deformation theory the thermodynamic stress τ_{33} is calculated using the complex formulae and the measured data. Dimensional changes are measured in the isothermal condition. The applied Cauchy stress σ_{33} is obtained from τ_{33} and fractional dimensional changes in lateral and vertical directions. Wave propagation is an adiabatic process that yields the adiabatic second order elastic constants. Third order elastic constants obtained from the wave speed data and the dimensional change are mixed elastic constants. These adiabatic and mixed elastic constants are converted into isothermal values using the thermodynamics of finite deformation of elastic solids developed by this author[1]. Then isothermal second order elastic compliance constants and isothermal third order elastic compliance constants are calculated. Finally applied load P is expressed in an elegant simple form as $P = A_a \tau_{33}$ [1 + $S^T_{33}\,\tau_{33}$ + (1/2) $(S^T_{333} - {S^T_{33}}^2)\,\tau^2_{33} + \cdots$], where A_a is the initial cross-sectional area of the specimen at zero load. For an isotropic specimen at zero load, $S^T_{33} = S^T_{11}$ and $S^T_{333} = S^T_{111}$. Several APLSG applied loads are obtained under several compressive loads up to 300 imperial tons at the Test Bay of Cornell University.

The APLSG presented in this article is a high capacity load gauge that covers the load range over 10 metric tons up to over a few thousands of metric tons. In this heavy load range it is impractical to compare the applied load with the dead weight, which yields directly the applied load in force unit of newton or kg. Most of the commercially available load cells are based on the some sorts of strain gages arrangement attached on the surface of a load carrying member and therefore the outputs of the load cells are an amplified electrical quantities in units of either volt, ampere, or ohm. These amplified outputs of the load cells are in some proportion to the applied load. Therefore, the strain-gage based load cells should be calibrated usually against applied known dead weight to know their real outputs. The strain-gage based load cells also drift with time and so they should be occasionally calibrated, which could be expensive and difficult in the force range higher than mega newton. Other type of load cells based on displacement, such as a proving ring and linear variable differential transformer, etc. need also be calibrated, because their outputs are not in units of force or mass. The load cells based on torque or moment applied on the load-carrying member also require calibration because of the same reason. To the author's knowledge, there is currently no load cell or load gauge that directly outputs the applied load in units of force or mass.

This paper addresses the short-comings of the currently available load cells in the higher load range by inventing a new load gauge that directly outputs the applied load in units of force or mass. In the past this author published several articles on the absolute stress gauge and acoustoelastcity.[1-4] This article is the extension of the author's past works. The basic idea is simple according to the linear elasticity theory. In the uniaxial homogeneous loading, say in the vertical direction 3, on an initially isotropic specimen at zero load, strain ε_{33} is linearly related to the Cauchy stress σ_{33} by the Hooke's law $\sigma_{33} = E\varepsilon_{33}$, where $E = {S_{33}}^{-1}$ is the Young's modulus of a load-carrying specimen and can be easily and accurately obtained by measuring the longitudinal and shear wave-speeds, which also yield the Poisson's ratio $\vartheta = -S_{13}{S_{33}}^{-1}$. For an isotropic solid, $S_{11} = S_{22} = S_{33}$ and $S_{12} = S_{13}$. Here, S_{ij} (i,j = 1,2, or 3) is the second order elastic compliance constants. Measuring the ε_{33} accurately in the loading direction is much more difficult than the horizontal strain ε_{11}or ε_{22}, which can be easily and accurately measured by measuring the dimensional change of the specimen in the horizontal direction. . Here, S_{11} and S_{33} are the elastic compliance constants referred to horizontal and vertical direction, respectively. Then vertical strain ε_{33} equal to $-\varepsilon_{11\vartheta^{-1}}$, when multiplied by E, yields the Cauchy stress σ_{33}. σ_{33} multiplied by the cross-sectional area of the specimen in situ finally yields the applied load.

However, a slight complication arises in this method. The Young's modulus and Poisson's ratio obtained from the longitudinal and shear wave-speeds are adiabatic constants, while the dimensional changes are measured in the isothermal condition. The adiabatic Young's modulus and Poisson's ratio can be easily converted into isothermal values by using the thermodynamics of elastic solids[1]. The isothermal Young's modulus and isothermal Poisson's ratio should be used to calculate the applied load. The applied load using the linear elasticity theory is fairly accurate within a few percent error but may not be accurate enough in most cases that require a higher accuracy. To improve the accuracy of the load measurement, the finite deformation theory of elastic solids is adopted to derive the formulas for the applied load. In finite deformation theory, the internal or mechanical energy contains not only harmonic potential but also anharmonic terms that contribute to the nonlinear elastic behavior of the material. The next section presents theoretical derivation of the formulas.

II. THEORY

Consider a load-carrying specimen that is loaded in the vertical direction 3. The Cauchy stress σ_{ij} (i, j = 1,2,3) applied in the vertical direction is specified by $\sigma_{ij} = \sigma_{33}\delta_{i3}\delta_{j3}$ and likewise the thermodynamic stress $\tau_{ij} = \tau_{33}\delta_{i3}\delta_{j3}$. The coordinates of a particle of the stressed body is said to be in the initial state and is denoted by the Cartesian coordinates **X.** The corresponding Cartesian coordinates under the stress-free zero load are denoted by vector **a**. As the stress level of the initial state **X** is arbitrary, it can include a stress free state **a** as a special case. For the sound wave propagating in the horizontal direction, say direction 1, with a wave normal **n =** [1,0,0], the Christoffel equation is expressed as[5]

$$\begin{pmatrix} C_{11}^{S}(\mathbf{X};\mathbf{X}) - \rho_X V^2 & 0 & 0 \\ 0 & C_{66}^{S}(\mathbf{X};\mathbf{X}) - \rho_X V^2 & 0 \\ 0 & 0 & C_{55}^{S}(\mathbf{X};\mathbf{X}) - \rho_X V^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \qquad (1)$$

where $C_{ij}^{S}(\mathbf{X};\mathbf{X})$ are the adiabatic thermodynamic elastic stiffness coefficients referenced to and evaluated at initial state **X,** ρ_X is the density of the material at the stressed initial state and V is the sound wave speed. The former **X** and the latter **X** inside the parenthesis represent an evaluation and reference states, respectively. When the reference and evaluation states are the same in the notation of the thermodynamic elastic coefficients, it is henceforth understood that the parenthesis of the thermodynamic elastic stiffness coefficients is denoted with the single argument, as in the following examples:

$$C_{ij}^{S\,or\,T}(\mathbf{X};\mathbf{X}) = C_{ij}^{S\,or\,T}(\mathbf{X}), \qquad C_{ijk}^{S\,or\,T}(\mathbf{X};\mathbf{X}) = C_{ijk}^{S\,or\,T}(\mathbf{X})$$

$$C_{ij}^{S\,or\,T}(\mathbf{a};\mathbf{a}) = C_{ij}^{S\,or\,T}(\mathbf{a}), \qquad C_{ijk}^{S\,or\,T}(\mathbf{a};\mathbf{a}) = C_{ijk}^{S\,or\,T}(\mathbf{a}).$$

The same convention applies to the compliance coefficients as

$$S_{ij}^{S\,or\,T}(\mathbf{X};\mathbf{X}) = S_{ij}^{S\,or\,T}(\mathbf{X}), \qquad S_{ijk}^{S\,or\,T}(\mathbf{X};\mathbf{X}) = S_{ijk}^{S\,or\,T}(\mathbf{X})$$

$$S_{ij}^{S\,or\,T}(\mathbf{a};\mathbf{a}) = S_{ij}^{S\,or\,T}(\mathbf{a}), \qquad S_{ijk}^{S\,or\,T}(\mathbf{a};\mathbf{a}) = S_{ijk}^{S\,or\,T}(\mathbf{a}),$$

where the superscripts S and T in the above equations denote adiabatic and isothermal process, respectively. The solution of Eq. (1) yields

$$\rho V_L^2(\mathbf{X}) = C_{11}^S(\mathbf{X}) \qquad \rho V_{21}^2(\mathbf{X}) = C_{66}^S(\mathbf{X}) \qquad \rho V_{31}^2(\mathbf{X}) = C_{55}^S(\mathbf{X}), \tag{2}$$

where $V_L(\mathbf{X}), V_{21}(\mathbf{X})$, and $V_{31}(\mathbf{X})$ denote velocities of the longitudinal wave, horizontally polarized shear (SH) wave in the direction 2 and vertically polarized shear (SV) wave in the direction 3, respectively, all propagating in the direction 1 and measured at the initial state **X.**

The dimensional change of the load-carrying member in the [100] direction is measured in an isothermal condition, while the elastic constants obtained from the wave speed measurements are adiabatic values. Isothermal elastic coefficients $C_{\alpha\beta}^T$ and $S_{\mu\nu}^T$ can be calculated from the adiabatic values by the following conversion formulae[1,6]:

$$C_{\alpha\beta}^T = C_{\alpha\beta}^S - T\left(\frac{\alpha_\mu^\tau\,\alpha_\nu^\tau\,C_{\mu\alpha}^T\,C_{\nu\beta}^T}{\rho_X C_\eta}\right) \qquad (\alpha, \beta = 1,2,\cdots 6) \tag{3a}$$

$$S_{\mu\nu}^T = S_{\mu\nu}^S + T\alpha_\mu^\tau\,\alpha_\nu^\tau\,/(\rho_X C_\tau) \qquad (\mu, \nu = 1,2,\cdots 6), \tag{3b}$$

$$\left[S_{\alpha\beta}^T\right] = \left[C_{\alpha\beta}^T\right]^{-1}, \tag{3c}$$

where $\alpha_\mu^\tau$are the thermal expansion coefficients at constant thermodynamic stress τ, T are is the absolute temperature, $S_{\mu\nu}^T$ and $S_{\mu\nu}^S$ are respectively isothermal and adiabatic thermodynamic elastic compliance coefficients, and $C_\eta$and C_τ are the specific heat at constant Lagrange strain η_{ij} and at constant thermodynamic stress τ_{ij}, respectively. The isothermal third order elastic constants are related to the mixed order constants $C_{\alpha\beta\gamma}^M \equiv \left(\partial C_{\alpha\beta}^S/\partial\eta_\gamma\right)_T$ by[7]

$$C^{T}_{\alpha\beta\gamma} = C^{M}_{\alpha\beta\gamma} - TC^{T}_{3\gamma}\ \frac{\partial}{\partial\tau_{33}}\left(\frac{\alpha^{\tau}_{\mu}\,\alpha^{\tau}_{\nu}\,C^{T}_{\mu\alpha}\,C^{T}_{\nu\beta}}{\rho_X C_{\eta}}\right) \tag{4a}$$

$$S^{T}_{\alpha\beta\gamma} = -S^{T}_{\alpha\nu}S^{T}_{\beta\mu}S^{T}_{\gamma\lambda}C^{T}_{\upsilon\mu\lambda}. \tag{4b}$$

When the direction of the applied load coincides with that of the principal strain or stress, it is convenient to introduce the principal stretches defined by

$$\frac{\partial x_i}{\partial a_j} = = \lambda_i\delta_{ij}\ (i \text{ fixed}). \tag{5}$$

Note that $\lambda_1 = \lambda_2$ and $\rho_X\ \rho_a{}^{-1} = \left(\lambda_1{}^2\lambda_3\right)^{-1}$apply to isotropic solids and also apply to cubic, hexagonal, and transversely isotropic solids when the applied loading direction coincides with one of cubic axes and the symmetry axis of hexagonal and transversely isotropic solids, respectively. For the case of $\sigma_{ij} = \sigma_{33}\delta_{i3}\delta_{j3}$ and $\tau_{ij} = \tau_{33}\delta_{i3}\delta_{j3}$

$$\lambda_1^2 = \lambda_2^2 = 1 + 2\eta_{11} = 1 + 2S^{T}_{13}\tau_{33} + S^{T}_{133}\tau^2_{33} + \cdots \tag{6a}$$

$$\lambda_3^2 = 1 + 2\eta_{33} = 1 + 2S^{T}_{33}\tau_{33} + S^{T}_{333}\tau^2_{33} + \cdots, \tag{6b}$$

where η_{11} and η_{33} are Lagrangian principal strains in the directions 1 and 3, respectively. Let L_{a1} and ΔL_{a1} denote the specimen length in horizontal direction 1at strain-free state and dimensional change in direction 1 under an applied load, respectively, $\lambda_1 = \lambda_2$ is given by

$$\lambda_1 = \lambda_2 = (L_{a1} + \Delta L_{a1})\ L_{a1}{}^{-1}. \tag{6c}$$

Denoting the Young's modulus of a specimen material to be $E(\mathbf{a})$ at strain-free natural state, note that for isotropic solids at zero load

$$S_{33}^T = S_{11}^T = S_{22}^T = E(\mathbf{a})^{-1} \qquad S_{13}^T = S_{12}^T = S_{23}^T \qquad S_{333}^T = S_{111}^T \,. \tag{6d}$$

Cauchy stress σ_{33} is related to thermodynamic stress τ_{33} by the Murnaghan's equation[7] as

$$\sigma_{33} = \frac{\rho_X}{\rho_a}\frac{\partial X_i}{\partial a_k}\frac{\partial X_j}{\partial a_l}\tau_{kl}\,\delta_{k3}\delta_{l3} = \lambda_1^{-2}\lambda_3\tau_{33}. \tag{7}$$

Now we introduce natural velocity W, which is defined by the original length at zero load L_{a1} in direction 1, divided by the travel time of the sound wave in situ. For the longitudinal waves

$$\rho_a W_L^2(\mathbf{X}) = \lambda_3\rho_X V_L^2(\mathbf{X}) = \lambda_3 C_{11}^S(\mathbf{X}) \tag{8a}$$

$$\lambda_3 C_{11}^T(\mathbf{X}) = C_{11}^T(\mathbf{a}) + \left[S_{12}^T(\mathbf{a})\left(2C_{11}^T(\mathbf{a}) + C_{111}^T(\mathbf{a}) + C_{112}^T(\mathbf{a})\right) + S_{11}^T(\mathbf{a})C_{112}^T(\mathbf{a})\right]\tau_{33} + \cdots. \tag{8b}$$

Making use of Eq. (3a) and $C_{11}^S(\mathbf{a};\mathbf{a}) = \rho_a W_L^2(\mathbf{a})$, it can be seen that

$$\rho_a\left(W_L^2(\mathbf{X}) - W_L^2(\mathbf{a})\right) + \frac{TC_{1\mu}^T C_{1\nu}^T \alpha_\mu^\tau \alpha_\nu^\tau}{\rho_a C_V}(\mathbf{a}) - \frac{\lambda_3\, TC_{1\mu}^T C_{1\nu}^T \alpha_\mu^\tau \alpha_\nu^\tau}{\rho_X C_\eta}(\mathbf{X}) =$$

$$\left[S_{12}^T(\mathbf{a})\left(2C_{11}^T(\mathbf{a}) + C_{111}^T(\mathbf{a}) + C_{112}^T(\mathbf{a})\right) + S_{11}^T(\mathbf{a})C_{112}^T(\mathbf{a})\right]\tau_{33} + \cdots. \tag{9a}$$

For shear waves, difference between isothermal and adiabatic values vanishes. Therefore,

$$\rho_a(W_{21}^2(\mathbf{X}) - W_{21}^2(\mathbf{a})) = \left[2S_{12}^T(\mathbf{a})\rho_a W_{21}^2(\mathbf{a}) + S_{12}^T(\mathbf{a})C_{111}^T(\mathbf{a})/2 + \left(S_{11}^T(\mathbf{a}) - S_{12}^T(\mathbf{a})\right)C_{112}^T(\mathbf{a})/2 - S_{11}^T(\mathbf{a})C_{123}^T(\mathbf{a})/2\right]\tau_{33} + \cdots \quad (9b)$$

$$\rho_a(W_{31}^2(\mathbf{X}) - W_{31}^2(\mathbf{a})) = \left[2S_{11}^T(\mathbf{a})\rho_a W_{31}^2(\mathbf{a}) + \left(S_{11}^T(\mathbf{a}) + S_{12}^T(\mathbf{a})\right)C_{111}^T(\mathbf{a})/4 - \left(S_{11}^T(\mathbf{a}) - S_{12}^T(\mathbf{a})\right)C_{112}^T(\mathbf{a})/4 - S_{12}^T(\mathbf{a})C_{123}^T(\mathbf{a})/2\right]\tau_{33} + \cdots \quad (9c)$$

The two terms in Eq. (8a) involving the thermal expansion coefficients and specific heats can be approximated to be linearly proportional toτ_{33}. We first notice that referring to Ref. 1, $C_\eta(\mathbf{X})$ is equal to $C_V(\mathbf{a})$ at a strain free state for isotropic and cubic solids and its change with strain or stress is negligible within the elastic limit of solids.

$$C_\eta(\mathbf{X}) = C_V(\mathbf{a}) + \left(\frac{\partial C_\eta(\mathbf{X})}{\partial \tau_{ij}}\right)_{T;\mathbf{a}} \tau_{ij} + \cdots \cong C_V(\mathbf{a}) \quad (10a)$$

$$\alpha_1^\tau(\mathbf{X}) = \alpha_2^\tau(\mathbf{X}) = \alpha^\tau(\mathbf{a}) + \left(\frac{\partial \alpha_1^\tau(\mathbf{X})}{\partial \tau_{33}}\right)_{T;\mathbf{a}} \tau_{33} + \cdots = \alpha^\tau(\mathbf{a}) + \left(\frac{\partial S_{12}^T(\mathbf{a})}{\partial T}\right)_{T;\mathbf{a}} \tau_{33} + \cdots \quad (10b)$$

$$\alpha_3^\tau(\mathbf{X}) = \alpha^\tau(\mathbf{a}) + \left(\frac{\partial \alpha_3^\tau(\mathbf{X})}{\partial \tau_{33}}\right)_{T;\mathbf{a}} \tau_{33} + \cdots = \alpha^\tau(\mathbf{a}) + \left(\frac{\partial S_{11}^T(\mathbf{a})}{\partial T}\right)_{T;\mathbf{a}} \tau_{33} + \cdots. \quad (10c)$$

$$\frac{T\alpha_\mu^\tau \alpha_\nu^\tau C_{1\mu}^\tau C_{1\nu}^\tau}{\rho_a c_V}(\mathbf{a}) = \frac{T}{\rho_a c_V(\mathbf{a})}\left[\alpha^\tau(\mathbf{a})\left(C_{11}^T(\mathbf{a}) + C_{12}^T(\mathbf{a}) + C_{13}^T(\mathbf{a})\right)\right]^2 = \frac{T\beta^2 {B^T}^2}{\rho_a c_V}(\mathbf{a}) \equiv \Delta, \quad (10d)$$

where $\beta = 3\alpha^{\tau}(\mathbf{a})$ is the volume thermal expansion coefficient and $B^T = 3^{-1}\big(C_{11}^T(\mathbf{a}) + 2C_{12}^T(\mathbf{a})\big)$ is the isothermal bulk modulus at zero load natural state **a.** For simplicity of notation, hence we drop the notation **(a)** when the physical variables are evaluated at zero load state natural state **a**. In Eqs. 10a-10d we use for specific heat and temperature coefficients of $S_{11}^T(\mathbf{a})$ and $S_{12}^T(\mathbf{a})$ the values quoted in literature.[8]

Letting

$$Z_0 \equiv \; 2S_{12}^T C_{11}^T + \; \Delta[2S_{12}^T C_{11}^T + (43^{-1})(S_{11}^T + 2S_{12}^T)C_{12}^T + 2\beta^{-1}C_{12}^T(\partial S_{11}^T/\partial T)_{\mathbf{a}} + 2\beta^{-1}(C_{11}^T + C_{12}^T)(\partial S_{12}^T/\partial T)_{\mathbf{a}}] \qquad (11a)$$

$$Z_1 \equiv S_{12}^T(1 + 3^{-1}2\Delta),\; Z_2 \equiv S_{11}^T(1 + 3^{-1}4\Delta) + S_{12}^T(1 + 3^{-1}8\Delta),\; Z_3 \equiv (3^{-1}2\Delta)(S_{11}^T + S_{12}^T), \qquad (11b)$$

Eq.8a, 8b, and 8c can be written as

$$Z_1 C_{111}^T + Z_2 C_{112}^T + Z_3 C_{123}^T = \; \rho_a\big({W_L}^2(\mathbf{X}) - {W_L}^2\big)\, {\tau_{33}}^{-1} \; - Z_0 \qquad (12a)$$

$$2^{-1}S_{12}^T C_{111}^T + 2^{-1}(S_{11}^T - S_{12}^T)C_{112}^T - 2^{-1}S_{11}^T C_{123}^T = \; \rho_a(W_{21}^2(\mathbf{X}) - W_{21}^2){\tau_{33}}^{-1} - 2S_{12}^T \rho_a W_{21}^2 \qquad (12b)$$

$$4^{-1}\;(S_{11}^T + S_{12}^T)C_{111}^T - 4^{-1}(S_{11}^T - S_{12}^T)C_{112}^T - 2^{-1}S_{11}^T C_{123}^T = \rho_a(W_{31}^2(\mathbf{X}) - W_{31}^2)\, {\tau_{33}}^{-1} - E_d - \; 2S_{11}^T \rho_a W_{31}^2 \,. \qquad (12c)$$

To express C_{111}^T, C_{112}^T and C_{123}^T in terms of τ_{33}, one calculates the following determinants:

$$D \equiv 8^{-1} \begin{vmatrix} Z_1 & Z_2 & Z_3 \\ S_{12}^T & S_{11}^T - S_{12}^T & -S_{11}^T \\ S_{11}^T + S_{12}^T & -S_{11}^T + S_{12}^T & -2S_{12}^T \end{vmatrix} \tag{13a}$$

$$E_a \equiv D^{-1}\rho_a \begin{vmatrix} W_L^2(\mathbf{X}) - W_L^2 & Z_2 & Z_3 \\ W_{21}^2(\mathbf{X}) - W_{21}^2 & 2^{-1}(S_{11}^T - S_{12}^T) & -2^{-1}S_{11}^T \\ W_{31}^2(\mathbf{X}) - W_{31}^2 & 4^{-1}(S_{12}^T - S_{11}^T) & -2S_{12}^T \end{vmatrix} \tag{13b}$$

$$E_b \equiv D^{-1}\rho_a \begin{vmatrix} Z_1 & W_L^2(\mathbf{X}) - W_L^2 & Z_3 \\ 2^{-1} S_{12}^T & W_{21}^2(\mathbf{X}) - W_{21}^2 & 2^{-1} S_{11}^T \\ 4^{-1}(S_{11}^T + S_{12}^T) & W_{31}^2(\mathbf{X}) - W_{31}^2 & -2^{-1}S_{12}^T \end{vmatrix} \tag{13c}$$

$$E_c \equiv D^{-1}\rho_a \begin{vmatrix} Z_1 & Z_2 & W_L^2(\mathbf{X}) - W_L^2 \\ 2^{-1}S_{12}^T & 2^{-1}(S_{11}^T - S_{12}^T) & W_{21}^2(\mathbf{X}) - W_{21}^2 \\ 4^{-1}(S_{11}^T + S_{12}^T) & 4^{-1}(S_{12}^T - S_{11}^T) & W_{31}^2(\mathbf{X}) - W_{31}^2 \end{vmatrix} \tag{13d}$$

$$E_d \equiv D^{-1}\rho_a \begin{vmatrix} Z_0/\rho_a & Z_2 & Z_3 \\ 2S_{12}^T W_{21}^2 & 2^{-1}(S_{11}^T - S_{12}^T) & -2^{-1}S_{11}^T \\ 2S_{11}^T W_{31}^2 & 4^{-1}(S_{12}^T - S_{11}^T) & -2^{-1}S_{12}^T \end{vmatrix} \tag{13e}$$

$$E_e \equiv D^{-1}\rho_a \begin{vmatrix} Z_1 & Z_0/\rho_a & Z_3 \\ 2^{-1}S_{12}^T & 2S_{12}^T W_{21}^2 & 2^{-1}S_{11}^T \\ 4^{-1}(S_{11}^T + S_{12}^T) & 2S_{11}^T W_{31}^2 & -2^{-1}S_{12}^T \end{vmatrix} \tag{13f}$$

$$E_f \equiv D^{-1}\rho_a \begin{vmatrix} Z_0 & Z_2 & Z_0/\rho_a \\ 2^{-1}S_{12}^T & 2^{-1}(S_{11}^T - S_{12}^T) & 2S_{12}^T W_{21}^2 \\ 4^{-1}(S_{11}^T + S_{12}^T) & 4^{-1}(S_{12}^T - S_{11}^T) & 2S_{11}^T W_{31}^2 \end{vmatrix}. \tag{13g}$$

Then,

$$C_{111}^T = E_a \, {\tau_{33}}^{-1} - E_d \,, \quad C_{112}^T = E_b \, {\tau_{33}}^{-1} - E_e, \quad C_{123}^T = E_c \, {\tau_{33}}^{-1} - E_f. \tag{14}$$

Note that all the physical variables appearing in Eqs. 13a-13g can be obtained from the four measured quantities as aforementioned with the thermal variables that can found in literature.

Using Eq. (4b), S_{133}^T in Eq. 6a can be expressed for an isotropic solid in terms of C_{ijk}^T. Then,

$$\eta_{11} = 2^{-1}(\lambda_1^2 - 1) = S_{12}^T\tau_{33} + 2^{-1}S_{112}^T\tau_{33}^2 + \cdots = S_{12}^T\tau_{33} - (gC_{111}^T + hC_{112}^T + 2gC_{123}^T)\tau_{33}^2 + \cdots = (S_{12}^T - gE_a - hE_b - 2gE_c)\tau_{33} + \left(gE_d + hE_e + 2gE_f\right)\tau_{33}^2 + \cdots \,, \tag{15}$$

where

$$g \equiv 2^{-1}S_{12}^T\left({S_{11}^T}^2 + {S_{12}^T}^2 + S_{11}^T S_{12}^T\right), \;\; h \equiv 2^{-1}\left({S_{11}^T}^3 + 3{S_{11}^T}^2 S_{12}^T + 9S_{11}^T {S_{12}^T}^2 + 5{S_{12}^T}^3\right) \tag{16}$$

The last equation of Eq. 15 is a quadratic equation of τ_{33}, which can be solved with measured λ_1 or η_{11}. When η_{11} is positive under a compressive load, the negative root of τ_{33} is taken by convention and the positive root of τ_{33} is taken by convention for the case of η_{11}being negative under a tensile load. $C_{111}^T, C_{112}^T,$ and C_{123}^T are then calculated via Eq. 14. $S_{333}^T = S_{111}^T$ for an isotropic solid is obtained using Eq. 4b. S_{111}^T is expressed as

$$S_{111}^{T} = -\left[\left({S_{11}^{T}}^{3} + 2{S_{12}^{T}}^{3}\right)C_{111}^{T} + 12gC_{112}^{T} + 6S_{11}^{T}{S_{12}^{T}}^{2}C_{123}^{T}\right]. \tag{17}$$

Finally, using Eqs. 7 and 6b, one obtains the applied load P on the load-carrying member of the APLSG as

$$\begin{aligned} P &= A_a\lambda_1^2\sigma_{33} = A_a\lambda_3\tau_{33} = A_a(1+2\eta_{33})^{1/2}\tau_{33} = A_a\tau_{33}(1+\eta_{33} - 2^{-1}\eta_{33}^2 + \cdots) \\ &= A_a\tau_{33}\left[1 + S_{33}^{T}\tau_{33} + 2^{-1}\left(S_{333}^{T} - {S_{11}^{T}}^{2}\right)\tau_{33}^2\right] + \cdots \\ &= A_a\tau_{33}[1 + E(\mathbf{a})^{-1}\tau_{33} + 2^{-1}(S_{333}^{T} - E(\mathbf{a})^{-2})\tau_{33}^2], \end{aligned} \tag{18}$$

where A_a and $E(\mathbf{a})$ are respectively the cross-sectional area and the Young's modulus of the specimen at zero load natural state. Note that for an isotropic solid, $S_{33}^{T} = S_{11}^{T}$, $S_{13}^{T} = S_{12}^{T}$, and $S_{333}^{T} = S_{111}^{T}$. (See Eq. 6d.).

Eq. (18) is an isothermal of equation of a solid that relates precisely finite non-hydrostatic uniaxial stress to finite uniaxial strain for the first time, because Eq. (18) takes care of the linear and nonlinear elastic contributions of material behavior under finite deformation and displacement η_{33} and Young's modulus $E(\mathbf{a})$ can be measured with a great accuracy as afore mentioned in Abstract

As will be shown in the appendix and in the section "IV. COMPRESSION TEST RESULTS AND DISCUSSION" in Part II of this article series, first term $A_a\tau_{33}$ in Eq. 18 contributes more than 99% to the applied load P and applied Cauchy stress σ_{33} in the APLSG. The second term $A_aE(\mathbf{a})^{-1}{\tau_{33}}^{2}$ contributes less than 1% in the APLSG, whereas the third term makes a negligible contribution.

III. CONCLUSIONS

1. This Part I theoretical paper derives a precise isothermal of equation of a solid (IOS) that relates finite non-hydrostatic uniaxial stress to finite uniaxial strain for the first time. The IOS is an important branch of materials science and applied physics.

2. This work demonstrates an APLSG that directly measures an applied stress in units of (mega) pascal and an applied load in absolute units of force or mass. In the higher load range over 10 metric tons, the APLSG has advantages over the conventional load cells that are based on the array of strain gages, the output of which is in units of electrical quantities in proportion to the applied load. Therefore, they require calibration usually against the dead weights. Calibration in range over mega-newton could be expensive and may be difficult to perform. Their output drift in time, requiring occasional calibrations.

3. The formula for the applied load is established with four measured quantities: three wave-speeds of the longitudinal, horizontally polarized shear (SH) and vertically polarized shear (SV) waves propagating in the direction perpendicular to the direction of the uniaxial applied load and a displacement of a load-carrying specimen normal to the loading direction. These four quantities can be measured with a great accuracy by adopting an up-to-date modern technique.

4. These four quantities measured both at zero load and at a loaded state may be used to determine the third order elastic stiffness and compliance constants. They can also determine the principal stretch in the loading direction and the density of specimen at a loaded specimen.

5. These four quantities also determine the thermodynamic and Cauchy stresses at a loaded state.

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APPENDIX

The outputs of Matlab software with data at zero load and about 200 imperial tons are shown as below:

EDU>> CalbFree_LoadCell

Enter the temperature and round-trip times at zero load

Enter the temperature in Celsius at zero load : 21.8

Enter the round-trip time in second of the longitudinal wave : 38.554e-06

Enter the round-trip time in second of the horizontally polarized shear wave : 78.194e-06

Enter the round-trip time in second of the vertically polarized shear wave : 79.206e-06

Enter the calculated values at zero load

rho_o = 2808.315658 kg/m^3; L_o = 0.121187 m; A_o = 0.011772 m^2

W_oL = 6286.636 m/s; W_o12 = 3099.662 m/s; W_o13 = 3060.058 m/s

C11S = 110.989682 GPa; C55S = 26.296947 GPa; C66S = 26.982031 GPa

C44S = 26.639489 GPa; C12S = 57.710704 GPa; B_S = 75.470364 GPa

Adiabatic-Isothermal Conversion Factor, Delta = 0.042943

Thermodynamic Grüneisen Parameter, gamma = 2.156941

C11T = 107.882216 GPa; C12T = 54.603238 GPa; C44T = 26.639489 GPa

S11T = 1.4048e-011 /(Pa); S12T = -4.7209e-012 /(Pa)

Isothermal Poissons ratio at zero load = 0.336050

Isothermal Youngs modulus at zero load = 71.183379 GPa

Zo = -1.2124e+000; Za = -4.8561e-012

Zb = 9.5911e-012; Zc = 2.6703e-013

g = -3.6190e-034; h = 1.1346e-033

Enter the round-trip times at load P

Enter the round-trip time of the longitudinal wave : 38.654e-06

Enter the round-trip time of the horizontally polarized shear wave : 78.402e-06

Enter the round-trip time of the vertically polarized shear wave : 78.746e-06

Enter the dimensional change of a specimen at load P : 8.825e-05

Measured L = 0.121276 m; Measured lamda1 = 1.0007282

W_L = 6270.373 m/s; W_12 = 3091.439 m/s; W_13 = 3077.934 m/s

tau33_1 = -155.722854 Mpa; tau33_2 = -68735.359409 MPa

C111T = -2.0156e+012; C112T = -5.0549e+011; C123T = -1.6502e+011

S111T = 3.2786e-021 S112T = -5.5071e-022

calculated lamda3 = 0.9978498

calculated lamda1 = 1.0007282

Summary of Input Data :

Round-trip time of Long. Wave: tau_oL = 3.855400e-005 sec; tau_L = 3.865400e-005 sec

Round-trip time of SH. Wave: tau_o12 = 7.819400e-005 sec; tau_12 = 7.840200e-005 sec

Round-trip time of SV. Wave: tau_o13 = 7.920600e-005 sec; tau_13 = 7.874600e-005 sec

L_o = 0.121187 m; L = 0.121276 m; lamda1 = 1.0007282

Display the values of Applied Stress and Load linear Cauchy Stress = -154.252287 MPa

Thermodynamic Stress tau33 = -155.722854 MPa

Cauchy Stress sigma33 = -155.161957 MPa

Applied Load P = -1.829179e+006 Newton

Applied Load P = -186.570928 metric Tons; Applied Load = -205.659242 imperial Tons